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NLS-MKdV Hierarchy and Its Hamiltonian Structures

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Abstract A new simple loop algebra is constructed, which is devote to establishing an isospectral problem. By making use of Tu scheme, NLS-MKdV hierarchy is obtained. Again via expanding the loop algebra above, another higher-dimensional loop algebra is presented. It follows that an integrable coupling of NLS-MkdV hierarchy is given. Also, the trace identity is extended to the quadratic-form identity and the Hamiltonian structures of the NLS-MKdV hierarchy and integrable coupling of NLS-MkdV hierarchy are obtained by the quadratic-form identity. The method can be used to produce the Hamiltonian structures of the other integrable couplings or multi-component hierarchies.

Keywords Hamiltonian structure · Integrable couplings · Quadratic-form identity

1 Introduction

The theory of integrable Hamiltonian systems with infinite dimensions has gone through rapid development since the late 1960s. In 1989, professor Tu proposed an efficient approach to searching for integrable Hamiltonian hierarchy. Nowadays, using Tu scheme, a lot of integrable systems with physics significance have been obtained [1–5]. At present, many integrable couplings of some integrable systems such as the TD hierarchy, Burgers hierarchy, etc. have been proposed [6, 7]. But Hamiltonian structures of integrable couplings systems haven't been worked out because the trace identity isn't suitable for them. In this paper, we extended the trace identity to be useful to the quadratic-form identity. As it application, the Hamiltonian structures of NLS-MKdV hierarchy and integrable couplings of NLS-MKdV hierarchy are worked out. The method proposed in this paper can be used generally.

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Set V_s to be an s-dimensional linear space with base $e_1, e_2, \ldots, e_s$, and let

$$a = \sum_{i=1}^{s} a_i e_i = (a_1, a_2, \ldots, a_s)^T, \qquad b = \sum_{i=1}^{s} b_i e_i = (b_1, b_2, \ldots, b_s)^T$$

be two elements of V_s and define a commutation operation as

$$[a, b] = \sum_{i,j=1}^{s} a_i b_j [e_i, e_j] = \sum_{i=1}^{s} c_i e_i = c = (c_1, c_2, \ldots, c_s)^T, \tag{1}$$

which makes V_s become a Lie algebra.

A corresponding loop algebra $\tilde{V}_s$ is presented with base and commutation operation respectively as follows

$$\begin{aligned} &e_i(m) = e_i \lambda^m, \quad [e_i(m), e_j(n)] = [e_i, e_j]\lambda^{m+n}, \\ &i, j = 1, 2, \ldots, s, \; m, n = 0, \pm 1, \pm 2, \ldots. \end{aligned} \tag{2}$$

The linear isospectral problem which is constructed by $\tilde{V}_s$ can be taken as:

$$\begin{cases} \psi_x = [U, \psi], & U, V, \psi \in \tilde{V}_s, \\ \psi_t = [V, \psi], & \lambda_t = 0. \end{cases} \tag{3}$$

From which, zero curvature equation

$$V_x = [U, V] \tag{4}$$

is easy to be derived, and we should define

$$\text{rank}(U) = \text{rank}\left(\frac{d}{dx}\right) = \alpha = const., \qquad \text{rank}(V) = \text{rank}\left(\frac{d}{dt}\right) = \xi = const. \tag{5}$$

Let the two arbitrary solutions V_1 and V_2 of (4) with the same rank have a linear relation

$$V_1 = \gamma V_2, \quad \gamma = const. \tag{6}$$

For $a, b \in \tilde{V}_s$, s-order matrix $R(b)$ is determined by

$$[a, b]^T = a^T R(b) \tag{7}$$

and constant matrix $F = (f_{ij})_{s\times s}$, is determined by

$$F = F^T, \qquad R(b)F = -(R(b)F)^T. \tag{8}$$

We introduce quadratic-form identity functional

$$\{a, b\} = a^T F b, \quad a, b \in \tilde{V}_s, \tag{9}$$

and consider functional

$$W = \{V, U_\lambda\} + \{\Lambda, V_x - [U, V]\}, \quad U, V, \Lambda \in \tilde{V}_s, \tag{10}$$

which has the following variation constraint conditions

$$\nabla_\Lambda W = V_x - [U, V] = 0, \tag{11}$$

$$\nabla_V W = U_\lambda - \Lambda_x + [U, \Lambda] = 0. \tag{12}$$

From (8), (9) and (7), we obtain

$$[\Lambda, V] - V_\lambda = \frac{\gamma}{\lambda} V, \quad \gamma = const. \tag{13}$$

then we obtain

$$\frac{\delta}{\delta u_i}\{V, U_\lambda\} = \lambda^{-\gamma} \frac{\partial}{\partial \lambda}\left(\lambda^r \left\{V, \frac{\partial U}{\partial u_i}\right\}\right), \quad 1 \le i \le l, \tag{14}$$

where γ is a constant to be determined. We call (14) the quadratic-form identity.

2 NLS-MkdV Hierarchy and Its Hamiltonian Structure

Now we take a special linear space

$$G_4 = \{a = (a_1, a_2, a_3, a_4)^T\}$$

with the commuting relations

$$\begin{aligned}[a, b] = (&2a_3b_4 - 2a_4b_3, 0, 2a_2b_4 - 4a_4b_1 - 2a_4b_2 \\ &+ 4a_1b_4, 4a_1b_3 + 2a_2b_3 - 4a_3b_1 - 2a_3b_2)^T.\end{aligned} \tag{15}$$

It is easy to verify that G_4 is a Lie algebra, which derives the resulting loop algebra

$$\tilde{G}_4 = \{a(\lambda) = a\lambda^m, a \in G_4, [a\lambda^m, b\lambda^n] = [a, b]\lambda^{m+n}, a, b \in G_4\}. \tag{16}$$

Taking the following isospectral problem

$$\begin{cases} \psi_x = [U, \psi], & U, V, \psi \in \tilde{G}_4, \\ \psi_t = [V, \psi], & \lambda_t = 0, \end{cases} \tag{17}$$

where

$$U = (0, \lambda, q, r)^T, \qquad V = (v_1, 0, v_3, v_4)^T, \quad v_k = \sum_{m \ge 0} v_{k,m}\lambda^{-m}.$$

Solving

$$V_x = [U, V] \tag{18}$$

gives rise to

$$\begin{cases} v_{1,mx} = -2rv_{3,m} + 2qv_{4,m}, \\ v_{3,mx} = 2v_{4,m+1} - 4rv_{1,m}, \\ v_{4,mx} = 2v_{3,m+1} - 4qv_{1,m}, \\ v_{1,0} = \beta = const., \quad v_{3,0} = v_{4,0} = 0, \quad v_{1,1} = 0, \quad v_{3,1} = 2\beta q, \\ v_{4,1} = 2\beta r, \quad v_{1,2} = \beta(q^2 - r^2), \quad v_{3,2} = \beta r_x, \quad v_{4,2} = \beta q_x, \\ v_{1,3} = \beta(q_x r - q r_x), \quad v_{3,3} = \frac{\beta}{2} q_{xx} + 2\beta q(q^2 - r^2), \\ v_{4,3} = \frac{\beta}{2} r_{xx} + 2\beta r(q^2 - r^2), \end{cases} \tag{19}$$

and

$$\text{rank}(v_{k,m}) = m \quad (k = 1, 3, 4, m \geq 0) \qquad \text{rank}(V) = \text{rank}\left(\frac{d}{dt}\right) = 0. \tag{20}$$

For arbitrary natural number n, denoting

$$V_+^{(n)} = \sum_{m=0}^{n} \begin{pmatrix} v_{1,m} \\ 0 \\ v_{3,m} \\ v_{4,m} \end{pmatrix} \lambda^{n-m}, \qquad V_-^{(n)} = \lambda^n V - V_+^{(n)},$$

then (18) can be written as

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = V_{-x}^{(n)} - [U, V_-^{(n)}], \tag{21}$$

the terms on the left-hand side of (21) are of degree ≥ 0, while the terms on the right-hand side of (21) are of degree ≤ 0. Therefore,

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = -\begin{pmatrix} 0 \\ 0 \\ 2v_{4,n+1} \\ 2v_{3,n+1} \end{pmatrix}.$$

Denoting $V^{(n)} = V_+^{(n)}$, the following zero-curvature equation

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0 \tag{22}$$

gives the integrable system

$$\begin{aligned} u_t = \begin{pmatrix} q \\ r \end{pmatrix}_t &= \begin{pmatrix} 2v_{4,n+1} \\ 2v_{3,n+1} \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} v_{3,n+1} \\ -v_{4,n+1} \end{pmatrix} = J \begin{pmatrix} v_{3,n+1} \\ -v_{4,n+1} \end{pmatrix} \\ &= J \begin{pmatrix} -4q\partial^{-1}r & -\frac{1}{2}\partial - 4q\partial^{-1}q \\ -\frac{1}{2}\partial + 4r\partial^{-1}r & 4r\partial^{-1}q \end{pmatrix} \begin{pmatrix} v_{3,n} \\ -v_{4,n} \end{pmatrix} = JL \begin{pmatrix} v_{3,n} \\ -v_{4,n} \end{pmatrix}, \end{aligned} \tag{23}$$

where J is a Hamiltonian operator and $JL = L^* J$. When taking $n = 2$, (23) was reduced to

$$\begin{cases} q_t = \beta r_{xx} + 4\beta r(q^2 - r^2), \\ r_t = \beta q_{xx} + 4\beta q(q^2 - r^2). \end{cases} \tag{24}$$

Letting $\beta = i, r, q = iq_1$ are real functions and denoting $Q = r + q = r + iq_1$, where $i = \sqrt{-1}$.

From (24), we have the well-known nonlinear Schrödinger equation (NLS)

$$iQ_t + Q_{xx} + 4Q|Q|^2 = 0. \tag{25}$$

When taking $n = 3$, (23) was reduced to

$$\begin{cases} q_t = \frac{\beta}{2} q_{xxx} + 6\beta q_x(q^2 - r^2), \\ r_t = \frac{\beta}{2} r_{xxx} + 6\beta r_x(q^2 - r^2). \end{cases} \tag{26}$$

Letting $q = ir$, (26) was reduced to M-KdV equation

$$r_t = \frac{\beta}{2} r_{xxx} - 12\beta r^2 r_x. \tag{27}$$

We call (23) NLS-MKdV equation because it includes NLS (25) and M-KdV (27). From (15), we have

$$[a,b]^T = a^T \begin{pmatrix} 0 & 0 & 4b_4 & 4b_3 \\ 0 & 0 & 2b_4 & 2b_3 \\ 2b_4 & 0 & 0 & -4b_1 - 2b_2 \\ -2b_3 & 0 & -4b_1 - 2b_2 & 0 \end{pmatrix} = a^T R(b). \tag{28}$$

Solving the matrix equation (8) for F yields

$$F = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{29}$$

$$\{a,b\} = (2a_1 + a_2)b_1 + (a_1 + a_2)b_2 - a_3 b_3 + a_4 b_4. \tag{30}$$

Hence, we deduce by the quadratic-form identity

$$\frac{\delta}{\delta u_i}(v_1) = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \begin{pmatrix} -v_3 \\ v_4 \end{pmatrix}, \tag{31}$$

where $V_k = \sum_{m \geq 0} V_{k,m} \lambda^{-m}$.

Comparing the coefficients of λ^{-n-1} yields

$$\frac{\delta}{\delta u} v_{1,n+1} = (\gamma - n) \begin{pmatrix} v_{3,n} \\ -v_{4,n} \end{pmatrix}.$$

Taking $n = 1$ in above equation gives $\gamma = 0$. Thus, Hamiltonian structure of (23) was derived as following

$$u_t = \begin{pmatrix} q \\ r \end{pmatrix}_t = J \begin{pmatrix} \frac{\delta}{\delta q} \\ \frac{\delta}{\delta r} \end{pmatrix} H_n = J \frac{\delta H_n}{\delta u}, \tag{32}$$

where $H_n = -\frac{1}{n} v_{1,n+1}$.

From (19), we obtain a recurrence operator

$$L = \begin{pmatrix} 4q\partial^{-1} r & -\frac{1}{2}\partial + 4q\partial^{-1} q \\ -\frac{1}{2}\partial - 4r\partial^{-1} r & -4r\partial^{-1} q \end{pmatrix},$$

which meets

$$\begin{pmatrix} v_{3,n+1} \\ -v_{4,n+1} \end{pmatrix} = L \begin{pmatrix} v_{3,n} \\ -v_{4,n} \end{pmatrix}.$$

Therefore, (32) can be written as

$$u_t = \begin{pmatrix} q \\ r \end{pmatrix}_t = JL^n \begin{pmatrix} 2\beta q \\ -2\beta r \end{pmatrix} = J \begin{pmatrix} \frac{\delta}{\delta q} \\ \frac{\delta}{\delta r} \end{pmatrix} H_{n+1}, \tag{33}$$

where $H_{n+1} = -\frac{1}{n+1} v_{1,n+2}$.

We observe that $JL = L^* J$, Hamiltonian functions H_l $(l \geq 1)$ are involutive each other and each H_l $(l \geq 1)$ is the common conserved density of (33). Therefore, the NLS-MKdV hierarchy (33) is a Liouville integrable.

3 The Integrable Coupling of NLS-MKdV Hierarchy and Its Hamiltonian Structure

Considering linear space

$$G_8 = \{a = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, v_8)^T\},$$

with the commuting relations

$$[a, b]^T = \begin{pmatrix} 2a_3b_4 - 2a_4b_3 \\ 0 \\ (4a_1 + 2a_2)b_4 - a_4(4b_1 + 2b_2) \\ (4a_1 + 2a_2)b_3 - a_3(4b_1 + 2b_2) \\ 2a_3b_8 - 2a_4b_7 + 2a_7b_4 - 2a_8b_3 \\ 0 \\ (4a_1 + 2a_2)b_8 - a_4(4b_5 + 2b_6) + (4a_5 + 2a_6)b_4 - a_8(4b_1 + 2b_2) \\ (4a_1 + 2a_2)b_7 - a_3(4b_5 + 2b_6) + (4a_5 + 2a_6)b_3 - a_7(4b_1 + 2b_2) \end{pmatrix}. \tag{34}$$

It is easy to verify that G_8 is a Lie algebra, which derives the resulting loop algebra

$$\tilde{G}_8 = \{a(\lambda) = a\lambda^m, a \in G_8, [a\lambda^m, b\lambda^n] = [a, b]\lambda^{m+n}, a, b \in G_8\}. \tag{35}$$

Taking the following isospectral problem

$$\begin{cases} \psi_x = [U, \psi], \ U, V, \psi \in \tilde{G}_8, \\ \psi_t = [V, \psi], \lambda_t = 0, \end{cases} \tag{36}$$

where $U = (0, \lambda, u_1, u_2, 0, 0, u_3, u_4)^T$, $V = (v_1, 0, v_3, v_4, v_5, 0, v_7, v_8)^T$, $v_k = \sum_{m\geq 0} v_{k,m}\lambda^{-m}$. Solving

$$V_x = [U, V] \tag{37}$$

gives rise to

$$\begin{cases} v_{1,mx} = 2u_1 v_{4,m} - 2u_2 v_{3,m}, \\ v_{3,mx} = 2v_{4,m+1} - 4u_2 v_{1,m}, \\ v_{4,mx} = 2v_{3,m+1} - 4u_1 v_{1,m}, \\ v_{5,mx} = 2u_1 v_{8,m} - 2u_2 v_{7,m} + 2u_3 v_{4,m} - 2u_4 v_{3,m}, \\ v_{7,mx} = 2v_{8,m+1} - 4u_2 v_{5,m} - 4u_4 v_{1,m}, \\ v_{8,mx} = 2v_{7,m+1} - 4u_1 v_{5,m} - 4u_3 v_{1,m}, \\ v_{1,0} = \alpha, \quad v_{3,0} = v_{4,0} = v_{5,0} = v_{7,0} = v_{8,0} = 0, \end{cases} \tag{38}$$

and

$$\operatorname{rank}(v_{k,m}) = m \quad (k = 1, 3, 4, 5, 7, 8, m \geq 0) \qquad \operatorname{rank}(V) = \operatorname{rank}\left(\frac{d}{dt}\right) = 0. \tag{39}$$

For an arbitrary natural number n, denoting

$$V_+^{(n)} = \sum_{m=0}^{n} \begin{pmatrix} v_{1,m} \\ 0 \\ v_{3,m} \\ v_{4,m} \\ v_{5,m} \\ 0 \\ v_{7,m} \\ v_{8,m} \end{pmatrix} \lambda^{n-m}, \qquad V_-^{(n)} = \lambda^n V - V_+^{(n)},$$

then (37) can be written as

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = V_{-x}^{(n)} - [U, V_-^{(n)}], \tag{40}$$

which the terms on the left-hand side of (40) are of degree ≥ 0, the terms on the right-hand side are of degree ≤ 0. Therefore,

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = -\begin{pmatrix} 0 \\ 0 \\ 2v_{4,n+1} \\ 2v_{3,n+1} \\ 0 \\ 0 \\ 2v_{8,n+1} \\ 2v_{7,n+1} \end{pmatrix}.$$

Denoting $V^{(n)} = V_+^{(n)}$, then the following zero-curvature equation

$$u_t = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t = \begin{pmatrix} 2v_{4,n+1} \\ 2v_{3,n+1} \\ 2v_{8,n+1} \\ 2v_{7,n+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 \\ -2 & 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} -v_{7,n+1} - v_{3,n+1} \\ v_{8,n+1} + v_{4,n+1} \\ -v_{3,n+1} \\ v_{4,n+1} \end{pmatrix}$$

$$= J \begin{pmatrix} -v_{7,n+1} - v_{3,n+1} \\ v_{8,n+1} + v_{4,n+1} \\ -v_{3,n+1} \\ v_{4,n+1} \end{pmatrix}, \tag{41}$$

where J is a Hamiltonian operator. From (34), we have

$$[a,b]^T = a^T \begin{pmatrix} 0 & 0 & 4b_4 & 4b_3 & 0 & 0 & 4b_8 & 4b_7 \\ 0 & 0 & 2b_4 & 2b_3 & 0 & 0 & 2b_8 & 2b_7 \\ 2b_4 & 0 & 0 & -4b_1-2b_2 & 2b_8 & 0 & 0 & -4b_5-2b_6 \\ -2b_3 & 0 & -4b_1-2b_2 & 0 & -2b_7 & 0 & -4b_5-2b_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4b_4 & 4b_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2b_4 & 2b_3 \\ 0 & 0 & 0 & 0 & 2b_4 & 0 & 0 & -4b_1-2b_2 \\ 0 & 0 & 0 & 0 & -2b_3 & 0 & -4b_1-2b_2 & 0 \end{pmatrix}$$

$$= a^T R(b). \tag{42}$$

Solving the matrix equation (8) for F yields

$$F = \begin{pmatrix} 2 & 1 & 0 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{43}$$

$$\begin{aligned}\{a,b\} &= (2a_1+a_2+2a_5+a_6)b_1 + (a_1+a_2+a_5+a_6)b_2 + (-a_3-a_7)b_3 \\ &\quad + (a_4+a_8)b_4 + (2a_1+a_2)b_5 + (a_1+a_2)b_6 - a_3b_7 + a_4b_8.\end{aligned}$$

Hence, we deduce by the quadratic-form identity

$$\frac{\delta}{\delta u_i}(v_1+v_5) = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma} \begin{pmatrix} -v_3-v_7 \\ v_4+v_8 \\ -v_3 \\ v_4 \end{pmatrix}, \tag{44}$$

where $V_k = \sum_{m\geq 0} V_{k,m}\lambda^{-m}$.

Comparing the coefficients of λ^{-n-1} yields

$$\frac{\delta}{\delta u}(v_{1,n+1}+v_{5,n+1}) = (\gamma-n) \begin{pmatrix} -v_{3,n}-v_{7,n} \\ v_{4,n}+v_{8,n} \\ -v_{3,n} \\ v_{4,n} \end{pmatrix}.$$

Taking $n=1$ in above equation gives $\gamma=0$. Thus, Hamiltonian structure of (42) was derived as following:

$$u_t = \begin{pmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \\ u_{4t} \end{pmatrix} = J \begin{pmatrix} \frac{\delta}{\delta u_1} \\ \frac{\delta}{\delta u_2} \\ \frac{\delta}{\delta u_3} \\ \frac{\delta}{\delta u_4} \end{pmatrix} H_n = J \frac{\delta H_n}{\delta u}, \tag{45}$$

where $H_n = -\frac{1}{n}(v_{1,n+1} + v_{5,n+1})$.

From (38), we obtain a recurrence operator

$$L = \begin{pmatrix} 4u_1\partial^{-1}u_2 & \frac{\partial}{2} + 4u_1\partial^{-1}u_1 & 4u_1\partial^{-1}u_4 + 4u_3\partial^{-1}u_2 & 4u_1\partial^{-1}u_3 + 4u_3\partial^{-1}u_1 \\ -\frac{\partial}{2} + 4u_2\partial^{-1}u_2 & 4u_2\partial^{-1}u_1 & 4u_2\partial^{-1}u_4 + 4u_4\partial^{-1}u_2 & 4u_2\partial^{-1}u_3 + 4u_4\partial^{-1}u_1 \\ 0 & 0 & 4u_1\partial^{-1}u_2 & \frac{1}{2}\partial + 4u_1\partial^{-1}u_1 \\ 0 & 0 & -\frac{1}{2}\partial + 4u_2\partial^{-1}u_2 & 4u_2\partial^{-1}u_1 \end{pmatrix},$$

which meets

$$\begin{pmatrix} -v_{7,n+1} - v_{3,n+1} \\ v_{8,n+1} + v_{4,n+1} \\ -v_{3,n+1} \\ v_{4,n+1} \end{pmatrix} = L \begin{pmatrix} -v_{3,n} - v_{7,n} \\ v_{4,n} + v_{8,n} \\ -v_{3,n} \\ v_{4,n} \end{pmatrix}.$$

Therefore, (41) can be written as

$$u_t = \begin{pmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \\ u_{4t} \end{pmatrix}_t = JL^n \begin{pmatrix} \alpha \\ -\alpha \\ -2\alpha u_3 \\ -2\alpha u_4 \end{pmatrix} = J \begin{pmatrix} \frac{\delta}{\delta u_1} \\ \frac{\delta}{\delta u_2} \\ \frac{\delta}{\delta u_3} \\ \frac{\delta}{\delta u_4} \end{pmatrix} H_{n+1}, \tag{46}$$

where $H_{n+1} = -\frac{1}{n}(v_{1,n+1} + v_{5,n+1})$.

When $u_1 = q, u_2 = r, u_3 = u_4 = 0$, (46) was reduced to the NLS-MKdV hierarchy (23). According to the definition of integrable coupling, (46) is the integrable coupling of (23).

Hamiltonian structures of integrable coupling of other hierarchies can be obtained by use of the quadratic-form identity which presented in this paper, such as AKNS hierarchy, KN hierarchy and so on. We will discuss in other paper.

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